Correlation of Friction Factors in Non-Newtonian Flow

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In Metzner and Reed's (2) recently published analysis of friction-factor data for non-Newtonian flow in cylindrical tubes the final correlation is based on an analytical solution of the equations of change for the laminar flow of a non-Newtonian fluid obeying the "power-law model." The following dimensional analysis approach gives an alternate method of interpreting the results of Metzner and Reed and may be of interest in connection with flow problems for other geometries and other non-Newtonian models.

The equations of change (1, 4) for the steady, isothermal flow of a pure incompressible fluid with no external body forces are

equation of continuity:

$$(\nabla \cdot \mathbf{v}) = 0 \qquad (1)$$

equation of motion:

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p - (\nabla \cdot \boldsymbol{\tau}) \qquad (2)$$

in which $\mathbf{v}$ is the local flow-velocity vector, ρ is the (constant) fluid density, p is the static pressure, and $\boldsymbol{\tau}$ is the shear-stress tensor. These equations are valid in both laminar and turbulent flow, and they may be applied to both Newtonian and non-Newtonian systems.

For the incompressible non-Newtonian power-law model $\boldsymbol{\tau}$ is given by (3)

$$|\, \boldsymbol{\tau} \,| = \beta \,|\, \nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger} \,|^{\alpha} \qquad (3)$$

in which $\dagger$ indicates the transpose of the dyadic $\nabla\mathbf{v}$, and β and α are the phenomenological constants (the latter denoting the degree of deviation from non-Newtonian behavior). The Newtonian fluid may be regarded as a special case of the empirical relation given in Equation (3) with $\beta = \mu$ and $\alpha = 1$.

The equations of change [with Equation (3) substituted into Equation (2)] may be written in dimensionless form by using two scale quantities: the diameter of the cylindrical tube D and the average linear-flow velocity V. Multiplication of Equation (1) by (D/V) and Equation (2) by $(D/\rho V^2)$ then gives

equation of continuity:

$$(\nabla^* \cdot \mathbf{v}^*) = 0 \qquad (4)$$

equation of motion:

$$(\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^* = -\nabla^* p^* - \left(\nabla^* \cdot \frac{1}{Re_\alpha} \,|\, \nabla^*\mathbf{v}^* + (\nabla^*\mathbf{v}^*)^{\dagger} \,|^{\alpha}\right) \qquad (5)$$

in which $\mathbf{v}^* = \mathbf{v}/V$, $\nabla^* = D\nabla$, and $p^* = p/\rho V^2$. A modified Reynolds number $Re_\alpha = D^\alpha V^{2-\alpha}\rho/\beta$ also appears quite naturally as a dimensionless group in the reduced equation. If the tube is considered to be of infinite length (this statement is tantamount to neglecting end effects) then the solution of these equations will give the reduced velocity $\mathbf{v}^*$ as a function of the reduced position coordinates $\mathbf{x}^*$

$$\mathbf{v}^* = \mathbf{v}^*(\mathbf{x}^*; Re_\alpha, \alpha) \qquad (6)$$

in which the modified Reynolds number Re_α and the exponent α appear parametrically.

The connection between the reduced velocity distribution [Equation (6)] and the friction factor must now be established. For the cylindrical tube of infinite length the friction factor is defined as

$$(\tau_{rz})_{wall} = (1/2)\rho V^2 \cdot f \qquad (7)$$

[This definition is consistent with the friction factor defined in Equation (9) of Metzner and Reed's paper.] According to Equation (3) the shear stress at the wall is also given by

$$(\tau_{rz})_{wall} = (-\partial v_z/\partial r)^{\alpha}{}_{wall} \qquad (8)$$

in which r is the radial coordinate and v_z is the component of $\mathbf{v}$ in the axial direction. From these two relations one obtains then:

$$\begin{aligned} f &= -(2\beta/\rho V^2)(\partial v_z/\partial r)^{\alpha}{}_{wall} \\ &= -(2/Re_\alpha)(\partial v^*_z/\partial r^*)^{\alpha}{}_{wall} \end{aligned} \qquad (9)$$

But since $\mathbf{v}$ has the dependence given in Equation (6), the dimensionless velocity gradient evaluated at the wall must depend only upon Re_α and α. Hence one obtains finally

$$f = f(Re_\alpha, \alpha) \qquad (10)$$

That is, one would expect the friction factor to depend upon two dimensionless quantities, Re_α and α. Dimensional analysis alone cannot give any additional information about the manner in which Re_α and α are combined.

For the *laminar region* the exact form for $f(Re_\alpha, \alpha)$ is known. The solution of the equation of motion gives for the velocity distribution (3)

$$v(r) = (\Delta p/2\beta L)^{1/\alpha}\left(1 + \frac{1}{\alpha}\right)^{-1} \cdot \left(R^{1+1/\alpha} - r^{1+1/\alpha}\right) \qquad (11)$$

in which $\Delta p/L$ is the pressure gradient through the tube and R is the tube radius. From this expression one may then get

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the power-law analogue of the Hagen-Poiseuille equation

$$Q = \pi R^2 V = 2\pi \int_0^R vr\,dr$$
$$= \pi\left(\frac{\Delta p}{2\beta L}\right)^{1/\alpha}\left(\frac{R^{3+1/\alpha}}{3 + 1/\alpha}\right) \quad (12)$$

The friction factor is defined in terms of the pressure drop and average flow velocity by

$$(\Delta p/L) = (1/2R)(\tfrac{1}{2}\rho V^2)(4f) \quad (13)$$

[which is the same as Metzner and Reed's Equation (9)]. Combination of Equations (12) and (13) gives then directly

$$f(Re_\alpha, \alpha) = 16\,\frac{\frac{1}{8}2^\alpha(3 + 1/\alpha)^\alpha}{Re_\alpha}$$
$$= \frac{16}{N_{Re}} \quad (14)$$

in which N_{Re} is the Reynolds number of Metzner and Reed. Hence the expression for N_{Re} can be obtained without any reference to the Rabinowitsch equation. It should be noted that N_{Re} is just a special combination of the Re_α and α predicted from dimensional analysis.

For the *turbulent region* the Metzner and Reed method of correlation inherently assumes that in the function $f(Re_\alpha, \alpha)$ the parameters Re_α and α will appear in the same special combination as in laminar flow—that is, f is a function of N_{Re} alone. The data available at the present time are probably not extensive enough to enable one to assess the correctness of the Metzner and Reed assumption. It should, however, be kept in mind that the power-law model is empirical; the fact that the flow curves (throughput vs. pressure drop) for laminar flow are described adequately by the *integrated* form of the power law in Equation (12) does not necessarily imply the correctness of the *differential* statement of the power law in Equation (3). Hence attempts to correlate the turbulent-flow region on the basis of the power-law model may prove impossible because of the inherent inadequacy of the model.

In the preparation of this note the author has had the benefit of correspondence with Professor Metzner and has profited considerably from the interchange of ideas.

LITERATURE CITED

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(*Dr. Metzner's reply appears on page 10S*)